Wave Propagation in Biaxial Planar Waveguides using Equivalent Circuit in Laplace Space

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Abstract:
By applying Laplace transformations to Maxwell's equations, transmission line equations are derived for guided wave propagation in planar waveguides. For the first time an intuitive equivalent T-circuit model is derived for inhomogeneous planar waveguides of materials having **biaxial** dielectric permittivity tensor, which may have **conductivity**, and **attenuation** along the propagation direction. The materials may also have **arbitrary refractive index profile along any of the three axes**. TE and TM mode equivalent circuits for dielectric layers are derived and the mode propagation constants are calculated from the resonances of the cascade of the elementary T-circuits.
The method can be extended, so as complete waveguide characterisation would be possible, allowing dispersion, cut-off frequencies and mode field plots to be calculated incorporating the influence of material conductivity and attenuation.

Keywords: Waveguides, Electromagnetic waves, Biaxial Crystals Integrated Optics,

1. Introduction:

Fourier transform techniques when applied to Maxwell's equations, allow the study of wave propagation in planar and cylindrical waveguides [1,2]. This leads to the concept of modelling electromagnetic phenomena as equivalent circuits offering strong electrical engineering intuitive undertones. The approach has also been applied to many other areas of electromagnetics including antenna theory [3], scattering phenomena [4] and recently in the field of Quantum Mechanics [5,6].

The concept of this work is based on modelling waveguides as simple electric circuits, such that the waveguide modal properties translate and can be derived from the resonant frequencies of the equivalent circuits. This is common sense in electrical engineering, and standard circuit theory can subsequently be applied to the circuits.

In this paper, a model is developped for wave propagation in planar inhomogeneous inisotropic biaxial materials. The model is general and accounts for the material's wave attenuation along the propagation direction, and includes the material conductivity. This model should be useful therefore to modelling waveguide layers realistically.

The need to incorporate wave attenuation and material conductivity in this model, as contrasted to previous work, lead us to applying Laplace transforms, instead of Fourier transforms.

Numerical results are presented for the variation of the normalised propagation constant of the fundamental TE and TM modes with attenuation and conductivity.

2. Maxwell's Equations in Laplace Space:

A homogeneous planar waveguiding layer with the corresponding reference coordinates is shown in Figure 1. Propagation os assumed along the z-axis. The y-invariance in this propagation problem leads to $\frac{\partial}{\partial y} = 0$ for the relevant electromagnetic fields.

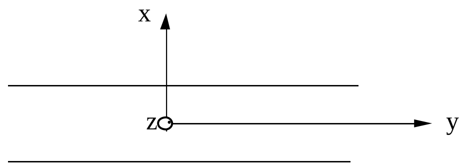

Figure 1: A thin waveguiding element with reference co-ordinates.

A uniform thin waveguiding layer can be described by its permittivity tensor, as follows:

$$\underline{\varepsilon} = \varepsilon_0 \begin{pmatrix} \varepsilon_1 \, 0 \, 0 \\ 0 \, \varepsilon_2 \, 0 \\ 0 \, 0 \, \varepsilon_3 \end{pmatrix}$$

The tensor nondiagonal elements are zero, and the tensor major axes aligned with the coordinate directions of Figure 1. The permeability of the material is assumed to be $\mu = \mu_0$, and σ , is the material conductivity.
The electric and magnetic fields E,H inside the layer are functions of (x,y,z) and time (t). The field components can in general be expressed by:

$$A(x, y, z, t) = \mathrm{Re}[\, A(x, y, z) e^{j\omega t} \,].$$

Hence the Laplace transform of the fields along the propagation direction is given by:

$\overset{0}{A}(z) = \int_0^{\infty} A(z) e^{-\gamma z} dz$, where $\gamma = \alpha - j\beta$. and α is the attenuation coefficient, and β is the mode propagation constant.

Maxwell's vector wave equations for time dependent fields are given by:

$$\left.\begin{aligned}\nabla\times \underline{E} &= -\dot{\underline{B}} = -m\dot{\underline{H}}\\ \nabla\times \underline{H} &= \underline{e}\,\dot{\underline{E}} + \underline{J}\end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots\ldots(1)$$

In Laplace space, the components of equation 1 reduce to:

$$\left.\begin{aligned} -g\,\overset{0}{E}_y &= -jwm\,\overset{0}{H}_x & -g\,\overset{0}{H}_y &= (jwe_1+s)\,\overset{0}{E}_x\\ -\frac{\P\,\overset{0}{E}_z}{\P x} + g\,\overset{0}{E}_x &= -jwm\,\overset{0}{H}_y & g\,\overset{0}{H}_x - \frac{\P\,\overset{0}{H}_z}{\P x} &= (jwe_2+s)\,\overset{0}{E}_y\\ \frac{\P\,\overset{0}{E}_y}{\P x} &= -jwm\,\overset{0}{H}_z & \frac{\P\,\overset{0}{H}_y}{\P x} &= (jwe_3+s)\,\overset{0}{E}_z \end{aligned}\right\} \quad ..(2)$$

The boundary conditions state that the field components across waveguide boundaries, are continuous, hence:

a) the tangential components of E_y, E_z, H_y, H_z are continous, and

b) the normal components of the vectors $\dot{B}$, *and* $J + e\,\dot{E}$ are continuous.

In Laplace space this means that the corresponding transformed fields are also continuous across charge or current free bandaries.

3. Equivalent Transmission Lines:

3.1 TM Modes:

Let us define new variables as follows:

$$\overset{0}{V}_E = -jg\,\overset{0}{E}_z$$

$$\overset{0}{I}_E = jg\,\overset{0}{H}_y = (we_1 - js)\,\overset{0}{E}_x$$

Using equations 2, the following transmission line equations can be derived:

$$\left.\begin{aligned}\frac{\P\,\overset{0}{V}_E}{\P x} &= \frac{-g_E^2\,\overset{0}{I}_E}{j(we_1 - js)}\\ \frac{\P\,\overset{0}{I}_E}{\P x} &= -j(we_3 - js)\,\overset{0}{V}_E\end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots(3)$$

where $g_E^2 = b^2 - a^2 - w^2 me_1 + j(2ab + wms)$. Equations (3) represent a transmission line model for an infinitecimally thin waveguiding layer, transversly to the propagation direction.

The characteristic impedance of the transmission line is:

$$\overset{0}{Z}_E = \frac{g_E}{j[w^2 e_1 e_3 + s^2 - jws(e_3 + e_1)]^{\frac{1}{2}}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4)$$

The planar layer of thickness d, can be represented by a T- equivalent circuit as shown in Figure 2.

The series and parallel elements of the circuit are given by:

$$\overset{0}{Z}_{S,E} = \overset{0}{Z}_E .\tanh(g_E .d/2),$$
$$\overset{0}{Z}_{P,E} = \overset{0}{Z}_E / \sinh(g_E .d)$$

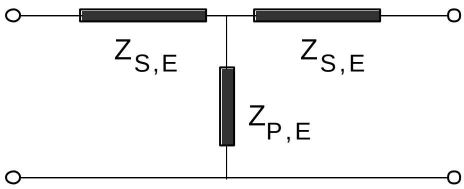

Figure 2: Equivalent T-circuits of TM modes of planar waveguide layer, of thickness d.

If the the layer is homogeneous, and of infinite thickness, such as a semi-infinite cladding layer, then it may be represented by its characteristic impedance $\overset{0}{Z}_E$, equation (4).

3.2. TE modes:

Similarly, for TE modes, let us define:

$$\overset{0}{V}_M = jg\,\overset{0}{H}_z$$
$$\overset{0}{I}_M = -wm_0 \overset{0}{H}_x = jg\,\overset{0}{E}_y$$

after some algebra the following transmission line equations can be derived representing an infinitesimally thin waveguiding layer:

$$\left.\begin{aligned} \frac{\P \overset{0}{V}_M}{\P x} &= \frac{-\overset{0}{I}_M g_M^2}{jwm_0} \\ \frac{\P \overset{0}{I}_M}{\P x} &= -jwm_0 \overset{0}{V}_M \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(5)$$

with $g_M^2 = b^2 - w^2 m_0 e_2 - a^2 + j(wm_0 s + 2ab)$.

The characteristic impedance $\overset{0}{Z}_M$ of equations (5) is given by:

$$\overset{0}{Z}_M = \frac{g_M}{jwm_0} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(6).$$

In a similar manner to TM modes the transmission line can be represented by an equivalent T-circuit as shown in Fig. 3, with series and parallel elements given by:

$$\overset{0}{Z}_{S,M} = \overset{0}{Z}_{M} \tanh(g_{M} d / 2)$$

$$\overset{0}{Z}_{P,M} = \overset{0}{Z}_{M} / \sinh(g_{M} d)$$

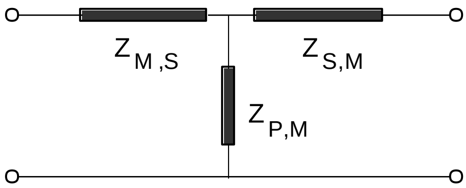

Figure 3: Equivalent T-circuit of TE modes of planar waveguide layer of thickness d.

4. Transverse Resonance of T-circuits:

An inhomogeneous waveguide can be represented as a set of thin homogeneous layers. This configuration can be represented as a sequence of equivalent circuits in tandem, as shown in Figure 4. The sequence is terminated by the characteristic impedance $\overset{0}{Z}_{T}$, and $\overset{0}{Z}_{B}$ of the top and base (semi-infinite) layers..

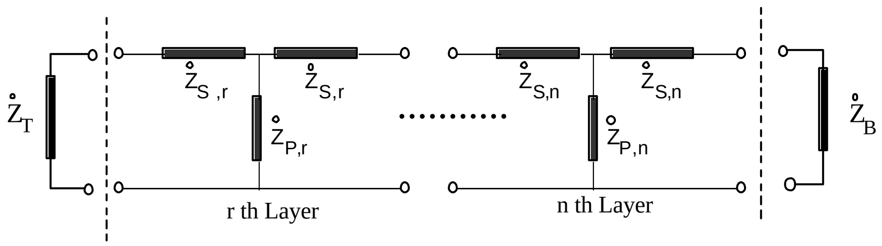

Figure 4: Cascade of equivalent circuits connected in tandem, representing inhomogeneous layers.

The propagation constant b , can be derived from the resonance frequencies of the above cascade. The impedance of the nth layer in terms of the impedance of the (n-1)th layer, the series and parallel Impedances of the T-circuits, may be expressed as follows:

$$\overset{0}{Z}_{n} = \frac{(\overset{0}{Z}_{n-1} + \overset{0}{Z}_{S,n})\,\overset{0}{Z}_{P,n}}{\overset{0}{Z}_{n-1} + \overset{0}{Z}_{S,n} + \overset{0}{Z}_{P,n}} + \overset{0}{Z}_{S,n} \quad \text{for n=1,2,3,4,........n} \qquad .....(7)$$

This recurrence relationship, allows us to calculate the total impedances up to a boundary (such as a well defined cladding layer) from either side of the boundary. Resonance occurs when at the arbitrary boundary, the impedance from one side, $\overset{0}{Z}_{1}$, and the other side, $\overset{0}{Z}_{2}$, are related as follows:

$$\mathrm{Im}(\overset{0}{Z}_{1} + \overset{0}{Z}_{2}) = 0 \qquad .... (8)$$

Using a root searching technique, varying b , the resonances can be located.

5. Cut-off frequencies of TE and TM modes.

Considering for simplicity a symmetrical waveguiding structure with substrate equal to superstrate refractive indicies, and the guiding layer being of a material with conductivity and attenuation such as: $n_1 k_0 > b > n_2 k_0$. The cut-of frequencies of the modes are obtained when $b = n_2 k_0$. With this condition, the propagation factors of equations (3) and (5) become:

$$g_{EC}^2 = n_2^2 k_c^2 - a^2 - w^2 m e_1 + j(2 a n_2 k_2 + w m s)$$

and

$$g_{MC}^2 = n_2^2 k_c^2 - a^2 - w^2 m e_2 + j(2 a n_2 k_2 + w m s)$$

respectively, for the TM and TE modes.

In this case, w becomes $w_c = k_c c$, where $c = \frac{1}{\sqrt{m_0 e_0}}$, $k_c = 2p / l_c$, , and l_c is the cut-off wavelength.

The cut-off wavelengths can be calculated in the same way as calculating the propagation constants. It is clear that both attenuation and conductivity affect the

6. Mode field plots:

The field plots of the propagation modes ban be plotted using this approach having first determined the propagation constant of the mode.
The proceedure can be illustrated with the aid of the following figure:

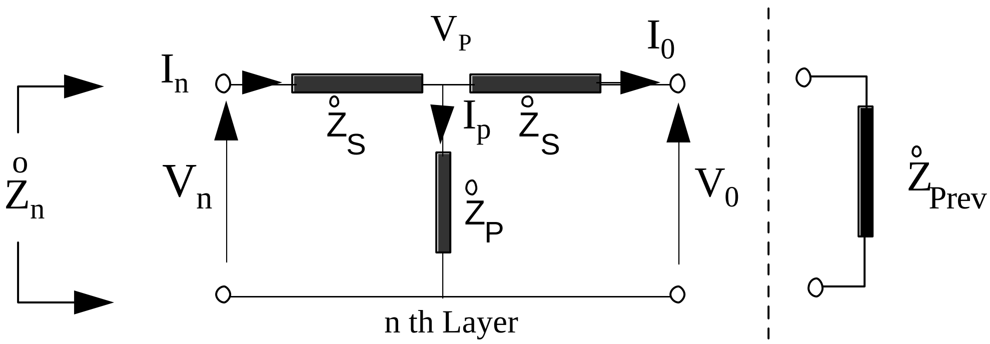

Figure 5: Voltages and currents through a representative transmission line.

From Figure 5, we can write for the voltage V_n , the "new" voltage (to the left of the transmission line) in terms of the Voltage at the node P , V_p , and the currents through node P as follows:

$$V_n = V_p + I_n \overset{0}{Z}_S$$

and

$$V_p = I_n \overset{o}{Z}$$

$$= I_n \frac{\overset{o}{Z}_p \left[\overset{o}{Z}_S + \overset{o}{Z}_{\Pr ev} \right]}{\overset{o}{Z}_P + \overset{o}{Z}_S + \overset{o}{Z}_{\Pr ev}}$$

with $\overset{o}{Z}_{\Pr ev} = \dfrac{V_o}{I_o}$ and $I_n = I_o \left[1 + \dfrac{\overset{o}{Z}_S}{\overset{o}{Z}_P} \right]$.

The above recerrence equations can be used to compute the field plots, starting from infinity, (a few core radii, when the field has decayed to zero), and successively calculating the field up to the core centre axis.
V_E *and* V_M can be computed with respect to the x axis of the waveguide.
E_x and E_y and hence be directly calculated and plotted.

7. Conclusions:

A simple transverse transmission line technique has been developped to competely characterise planar waveguide layers of arbitrary refractive index profile, and at the same time allowing the conductivity and attenuation of the material to be incorporated in the analysis.
Complete characterisation is possible, allowing the calculation of cut-off frequencies as well as mode field plots, under the influence of attenuation and material conductivity.

8. References:

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